Semester Two Examination, 2023

Question/Answer booklet

MATHEMATICS SPECIALIST UNITS 3&4

Section One: Calculator-free

SOLUTIONS

WA student number: In figures

In words ______________________________

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Your name ______________________________

Time allowed for this section

Reading time before commencing work: five minutes
Working time: fifty minutes

Number of additional answer booklets used (if applicable):

Materials required/recommended for this section

To be provided by the supervisor

This Question/Answer booklet
Formula sheet

To be provided by the candidate

Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: nil

Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised material. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of examination
Section One: Calculator-free	7	7	50	48	35
Section Two: Calculator-assumed	12	12	100	90	65
				Total	100

Instructions to candidates

1. The rules for the conduct of examinations are detailed in the school handbook. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer booklet preferably using a blue/black pen. Do not use erasable or gel pens.

3. You must be careful to confine your answers to the specific question asked and to follow any instructions that are specific to a particular question.

4. Show all your working clearly. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

5. It is recommended that you do not use pencil, except in diagrams.

6. Supplementary pages for planning/continuing your answers to questions are provided at the end of this Question/Answer booklet. If you use these pages to continue an answer, indicate at the original answer where the answer is continued, i.e. give the page number.

7. The Formula sheet is not to be handed in with your Question/Answer booklet.

Section One: Calculator-free **35% (48 Marks)**

This section has **seven** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time: 50 minutes.

Question 1 **(6 marks)**

(a) Determine $\frac{dy}{dx}$ when $x + y^3 - 6x^2y = 4$. (3 marks)

Solution
$1 + 3y^2y' - (12xy + 6x^2y') = 0$ $3y^2y' - 6x^2y' = 12xy - 1$ $\frac{dy}{dx} = \frac{12xy - 1}{3y^2 - 6x^2}$
Specific behaviours
✓ correctly differentiates y^3 term ✓ correctly differentiates x^2y term ✓ obtains correct derivative

(b) Determine the slope of the curve with equation $e^{6x} = 2 - 2\cos(2y)$ at the point $\left(0, \frac{\pi}{6}\right)$. (3 marks)

Solution
$6e^{6x} = 2\sin 2y \times 2y'$ $\left(0, \frac{\pi}{6}\right) \rightarrow 6 = 2\frac{\sqrt{3}}{2} \times 2y'$ $y' = \frac{6}{2\sqrt{3}} = \sqrt{3}$ Slope of curve is $\sqrt{3}$.
Specific behaviours
✓ correctly differentiates $\cos 2y$ term ✓ correctly substitutes ✓ correct slope

Question 2 **(6 marks)**

Function f is defined with domain $\{x \in \mathbb{R}: x > -2\}$ by $f(x) = \dfrac{3x-5}{x+2} = 3 - \dfrac{11}{x+2}$.

(a) Determine $f^{-1}\left(\dfrac{2}{3}\right)$. (2 marks)

Solution
$\dfrac{3x-5}{x+2} = \dfrac{2}{3}$ $9x - 15 = 2x + 4$ $7x = 19$ $x = \dfrac{19}{7}$ Hence $f^{-1}\left(\dfrac{2}{3}\right) = \dfrac{19}{7}$.
Specific behaviours
✓ reasonable attempt using any correct method ✓ correct value

Function g is defined with domain $\{x \in \mathbb{R}: 0 \le x \le 4\}$ by $g(x) = \sqrt{2x+1}$.

(b) State the range of $g^{-1}(x)$. (1 mark)

Solution
$0 \le g^{-1}(x) \le 4$ *Range of $g^{-1}(x)$ equals domain of $g(x)$*
Specific behaviours
✓ correct range, using any notation

(c) Determine the range of $f \circ g^{-1}(x)$. (3 marks)

Solution
Domain of f is range of g^{-1}. As x increases from 0 to 4, f is always increasing and so range will be from $f(0)$ to $f(4)$: $f(0) = -\dfrac{5}{2}, \qquad f(4) = \dfrac{7}{6}$ $-\dfrac{5}{2} \le f \circ g^{-1}(x) \le \dfrac{7}{6}$
Specific behaviours
✓ indicates f increasing over domain, or boundaries required ✓ obtains one boundary ✓ correctly writes range

Question 3 **(8 marks)**

The planes with equations $2x - y - z = 1, 2x + y + 2z = 6$ and $x + y + z = 2$ intersect at point P.

(a) Determine the coordinates of point P. (3 marks)

Solution
$\left.\begin{matrix} 2x - y - z = 1 \\ x + y + z = 2 \end{matrix}\right\} \rightarrow 3x = 3 \rightarrow x = 1$ $\left.\begin{matrix} 2x + 2y + 2z = 4 \\ 2x + y + 2z = 6 \end{matrix}\right\} \rightarrow y = -2$ $1 - 2 + z = 2 \rightarrow z = 3$ Hence $P(1, -2, 3)$.
Specific behaviours
✓ solves for one variable   ✓ solves for second variable   ✓ states correct coordinates

Points Q and R have coordinates $(-1, 8, -3)$ and $(-2, 10, -4)$ respectively.

(b) Determine the vector equation of the straight line through points Q and R in the form $\underset{\sim}{r} = \underset{\sim}{a} + \lambda \underset{\sim}{b}$. (2 marks)

Solution
$\underset{\sim}{b} = \begin{pmatrix} -1 \\ 8 \\ -3 \end{pmatrix} - \begin{pmatrix} -2 \\ 10 \\ -4 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$ $\underset{\sim}{r} = \begin{pmatrix} -1 \\ 8 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$
Specific behaviours
✓ obtains $\underset{\sim}{b}$ ✓ correct equation in required form

(c) Determine the Cartesian equation of the plane that contains point Q, point R and the origin. (3 marks)

Solution
$\underset{\sim}{n} = \begin{pmatrix} -1 \\ 8 \\ -3 \end{pmatrix} \times \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ -6 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ -1 \\ -3 \end{pmatrix}$ Hence equation of plane is $x - y - 3z = 0$.
Specific behaviours
✓ forms cross product using two vectors in plane   ✓ obtains normal   ✓ states correct equation

Question 4 **(7 marks)**

A small body moving in a straight line is initially at the origin O. t seconds later the body has displacement x metres and velocity v metres per second relative to O so that $v = 4e^{-0.2x}$.

(a) Determine the initial acceleration of the body. (2 marks)

Solution
$a = v\dfrac{dv}{dx} = 4e^{-0.2x}(-0.8e^{-0.2x})$ $t = 0, x = 0 \Rightarrow a(0) = -3.2\ \mathrm{m/s^2}$
Specific behaviours
✓ obtains correct expression for $a(x)$ ✓ correct initial acceleration

(b) Determine the displacement of the body as a function of time. (3 marks)

Solution
$v = \dfrac{dx}{dt} = 4e^{-0.2x}$ $\int e^{0.2x}\,dx = \int 4\,dt$ $5e^{0.2x} = 4t + c$ $t = 0, x = 0 \Rightarrow c = 5$ $e^{0.2x} = 0.8t + 1$ $0.2x = \ln(0.8t + 1)$ $x(t) = 5\ln(0.8t + 1)$
Specific behaviours
✓ formulates differential equation and separates variables   ✓ antidifferentiates, evaluates constant c ✓ correct displacement function

(c) Determine the velocity of the body after 5 seconds. (2 marks)

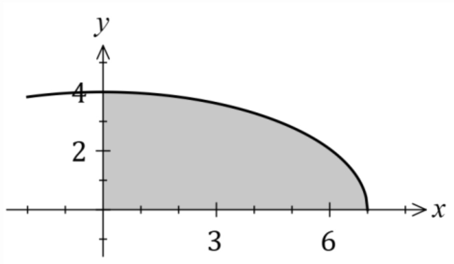
Solution
$v(t) = \dfrac{d}{dt}\big(x(t)\big) = \dfrac{5(0.8)}{0.8t + 1}$ $v(5) = \dfrac{4}{5} = 0.8\ \mathrm{m/s}$
Specific behaviours
✓ indicates appropriate method   ✓ correct velocity

Alternative Solution
$e^{0.2x} = 0.8(5) + 1 = 5$ $e^{-0.2x} = 0.2$ $v = 4(0.2) = 0.8\ \mathrm{m/s}$
Specific behaviours
✓ indicates appropriate method   ✓ correct velocity

Question 5 **(7 marks)**

The diagram shows part of the curve with equation

$$y = \frac{4}{7}\sqrt{7^2 - x^2}.$$



Use integration and the substitution $x = 7\sin\theta$ to determine the shaded area in the first quadrant bounded by $x = 0, y = 0$ and the curve.

Solution
$$A = \int_0^7 \frac{4}{7}\sqrt{7^2 - x^2}\,dx$$ $$x = 7\sin\theta \Rightarrow dx = 7\cos\theta\,d\theta$$ When $x = 0, \theta = 0$ and when $x = 7, \theta = \frac{\pi}{2}$. $$\begin{aligned} A &= \frac{4}{7}\int_0^{\frac{\pi}{2}} \sqrt{7^2 - 7^2\sin^2\theta} \times 7\cos\theta\,d\theta \\ &= 4\int_0^{\frac{\pi}{2}} 7\sqrt{1 - \sin^2\theta}\cos\theta\,d\theta \\ &= 28\int_0^{\frac{\pi}{2}} \cos^2\theta\,d\theta \\ &= 28\int_0^{\frac{\pi}{2}} \frac{1}{2}(1 + \cos 2\theta)\,d\theta \\ &= 14\left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{2}} \\ &= 14\left(\left[\frac{\pi}{2} + 0\right] - [0 + 0]\right) \\ &= 7\pi \end{aligned}$$
Specific behaviours
✓ writes integral in terms of x with correct bounds ✓ relates dx and $d\theta$ ✓ adjusts bounds in terms of θ ✓ writes and simplifies integral in terms of θ ✓ replaces $\cos^2\theta$ term using trig identity ✓ correctly antidifferentiates ✓ substitutes and simplifies to obtain area

Question 6 **(8 marks)**

Let $v = 1 - \sqrt{3}i$.

(a) Determine the three cube roots of v. (3 marks)

Solution
$v = 2\operatorname{cis}\left(-\frac{\pi}{3}\right)$ $v^{\frac{1}{3}} = 2^{\frac{1}{3}}\operatorname{cis}\left(-\frac{\pi}{9} + \frac{2n\pi}{3}\right)$ Hence the cube roots are: $\sqrt[3]{2}\operatorname{cis}\left(\frac{5\pi}{9}\right)$, $\sqrt[3]{2}\operatorname{cis}\left(-\frac{\pi}{9}\right)$, $\sqrt[3]{2}\operatorname{cis}\left(-\frac{7\pi}{9}\right)$
Specific behaviours
✓ writes v in polar form ✓ obtains one correct root ✓ correctly states all roots

(b) Consider the polynomial $P(z) = z^4 + 2z^3 + 2z^2 + kz + 24$, where k is a real constant.

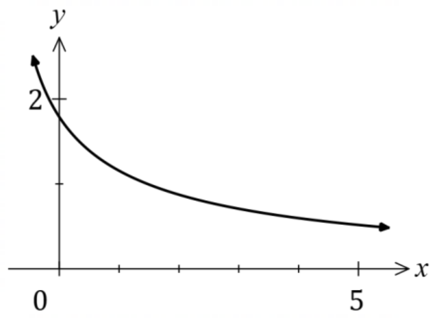
Given that $P(v) = 0$, solve the equation $P(z) = 0$. (5 marks)

Solution
P has real coefficients and so v and $\bar{v}$ are factors: $(z - v)(z - \bar{v}) = (z - 1 + \sqrt{3}i)(z - 1 - \sqrt{3}i)$ $= z^2 - 2z + 4$ Hence $z^4 + 2z^3 + 2z^2 + kz + 24 = (z^2 - 2z + 4)(z^2 + az + 6)$, where a is a real constant. Comparing coefficients of z^3 then $2 = a - 2 \Rightarrow a = 4$. Zeros of second quadratic factor: $z^2 + 4z + 6 = 0$ $(z + 2)^2 = -2 = 2i^2$ $z = -2 \pm \sqrt{2}i$ Solutions are $z = 1 \pm \sqrt{3}i, = -2 \pm \sqrt{2}i$. *Note that it is possible to deduce $k = 4$, but this is not required.*
Specific behaviours
✓ indicates $P(\bar{v}) = 0$ or $z - \bar{v}$ is a factor of P ✓ correctly determines quadratic factor of P ✓ determines second quadratic factor ✓ obtains a zero from second quadratic factor ✓ states all solutions

Question 7 **(6 marks)**

The graph of $y = \dfrac{4}{\sqrt{x^2 + 6x + 5}}$ is shown.

Determine the volume of the solid of revolution formed when the part of the curve between $x = 0$ and $x = 5$ is rotated about the x-axis.



Solution
$V = \int_0^5 \pi y^2 \, dx$ $= \pi \int_0^5 \frac{16}{x^2 + 6x + 5} dx$ $\frac{16}{x^2 + 6x + 5} = \frac{16}{(x+1)(x+5)} = \frac{a}{x+1} + \frac{b}{x+5}$ $x = -1, a = 16 \div (-1 + 5) = 4, \qquad x = -5, b = 16 \div (-5 + 1) = -4$ $V = 4\pi \int_0^5 \frac{1}{x+1} - \frac{1}{x+5} dx$ $= 4\pi[\ln(x+1) - \ln(x+5)]_0^5$ $= 4\pi \left[\ln\left(\frac{x+1}{x+5}\right)\right]_0^5$ $= 4\pi \left[\ln\left(\frac{6}{10}\right) - \ln\left(\frac{1}{5}\right)\right]$ $= 4\pi \ln 3$
Specific behaviours
✓ forms correct integral for volume   ✓ indicates need for partial fractions and factorises denominator   ✓ obtains partial fractions   ✓ rewrites integral using partial fractions   ✓ antidifferentiates and substitutes   ✓ correct volume, simplified

End of questions

Supplementary page

Question number: _________

Supplementary page

Question number: _________